A self-consistent model of evaporating BH's

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Y. Matsuo, Y. Yokokura and HK [arXiv:1302.4733]
Y. Yokokura and HK [arXiv:1409.5784, 1509.08472]

See also Pei-Ming Ho [arXiv:1505.02468, 1510.07157].

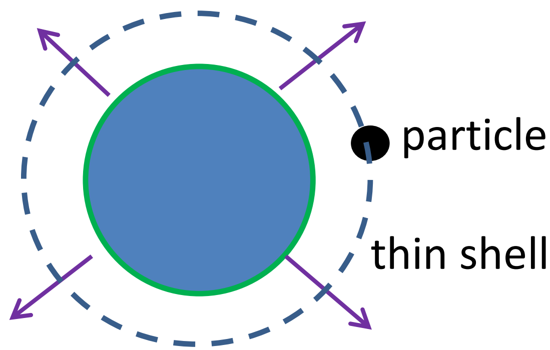
Nowadays everybody has heard about black hole evaporation.

# of theories = # of physicists

The basic idea

We start with the fact that black hole is to evaporate.

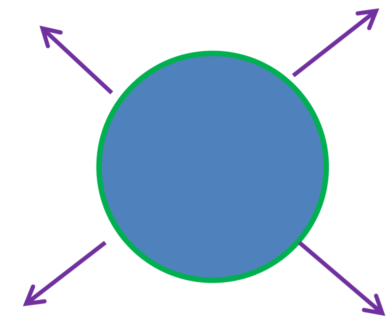
First we consider a spherically symmetric evaporating BH, and see what happens if we add a particle or a thin shell.



The crucial point is that the particle or the shell will never reaches the horizon of the BH.

This is because while the particle comes close to the horizon in a time scale of the Schwarzschild radius, the radius shrinks by the Hawking radiation. Therefore the particle will never catch up with the horizon.

More precisely, we assume the outside metric of the evaporating BH is given by the Schwarzschild like metric.



$$\boxed{\begin{gathered} ds^2 = -\frac{r-a(t)}{r}dt^2 + \frac{r}{r-a(t)}dr^2 + r^2 d\Omega^2. \\ \frac{da}{dt} = -\frac{2\sigma(a(t))}{a(t)^2} \end{gathered}}$$

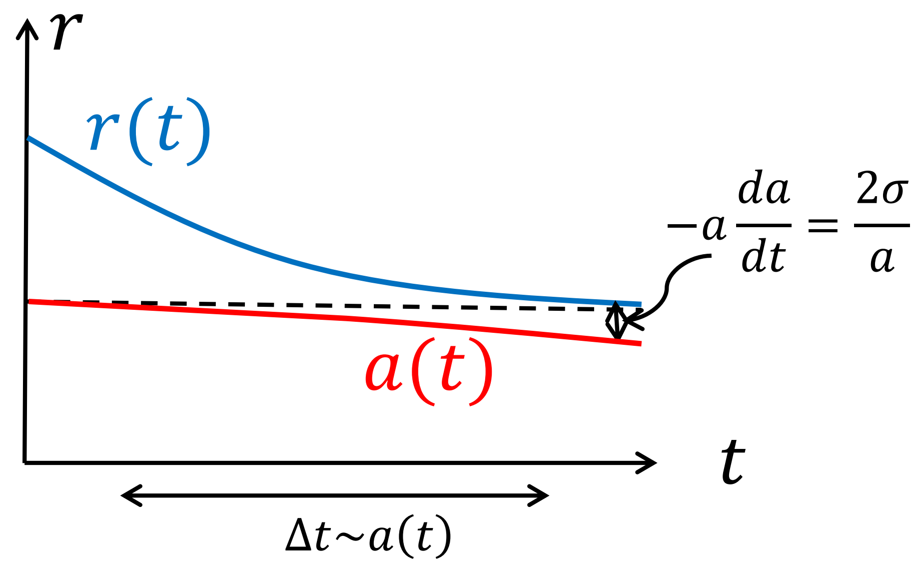
$$a(t) \equiv 2GM(t)$$

Then, the position $r(t)$ of the particle near $a(t)$ is determined by

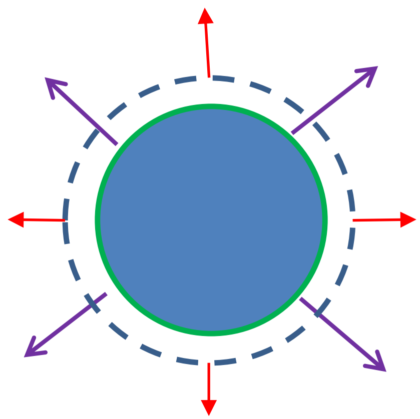
$$\frac{dr(t)}{dt} = -\frac{r(t)-a(t)}{r(t)}.$$

$r(t)$ does not completely catch up with $a(t)$ but approaches

$$R(a(t)) \equiv a(t) + \frac{2\sigma(a(t))}{a(t)}.$$



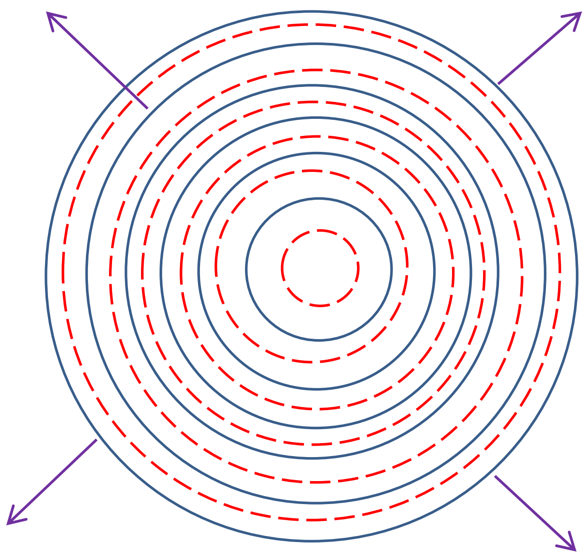
After the shell approaches to $R(a(t))$ in a time scale of $a(t)$, the total system looks like a BH with mass $m + \Delta m$, where Δm is the mass of the added shell.



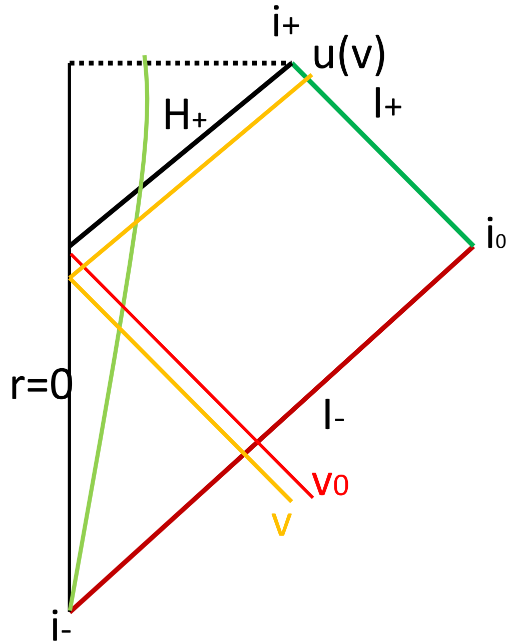
Now the radiation of the total system consists of two parts; one is from the inside BH with a slight redshift due to the outer shell, and the other is the radiation created by the outer shell.
We can show that the total amount of radiation is precisely the same as that of the BH with mass $m + \Delta m$.

By using this procedure recursively, we can imagine that BH consists of many shells.

If we see the system from outside, it looks like an ordinary evaporating BH, but it has a well defined finite metric in the entire region.



Introduction (1)



Usually Hawking radiation is considered in the following manner:

(1) Matter collapses to form a black hole.
(2) Radiation appears in the BH metric.
(3) The system loses energy, becomes smaller and finally disappears because of the radiation.

Then we have the following questions:

(1) What happens to the singularity?
(2) What happens to the information that fell into the BH?

⇒In order to answer these questions, we should solve the whole system including the back reaction from the Hawking radiation.

As a step towards this direction, we try to solve the self-consistent equation:

$$G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle\,.$$

Introduction (2) : self-consistent eq.

In this self-consistent equation

$$G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle \ ,$$

we treat the metric and the radiation as follows:

(1) The metric $g_{\mu\nu}$ is regarded as a classical field, and determined by the Einstein equation with the expectation value of the energy momentum tensor $\langle T_{\mu\nu}\rangle$.

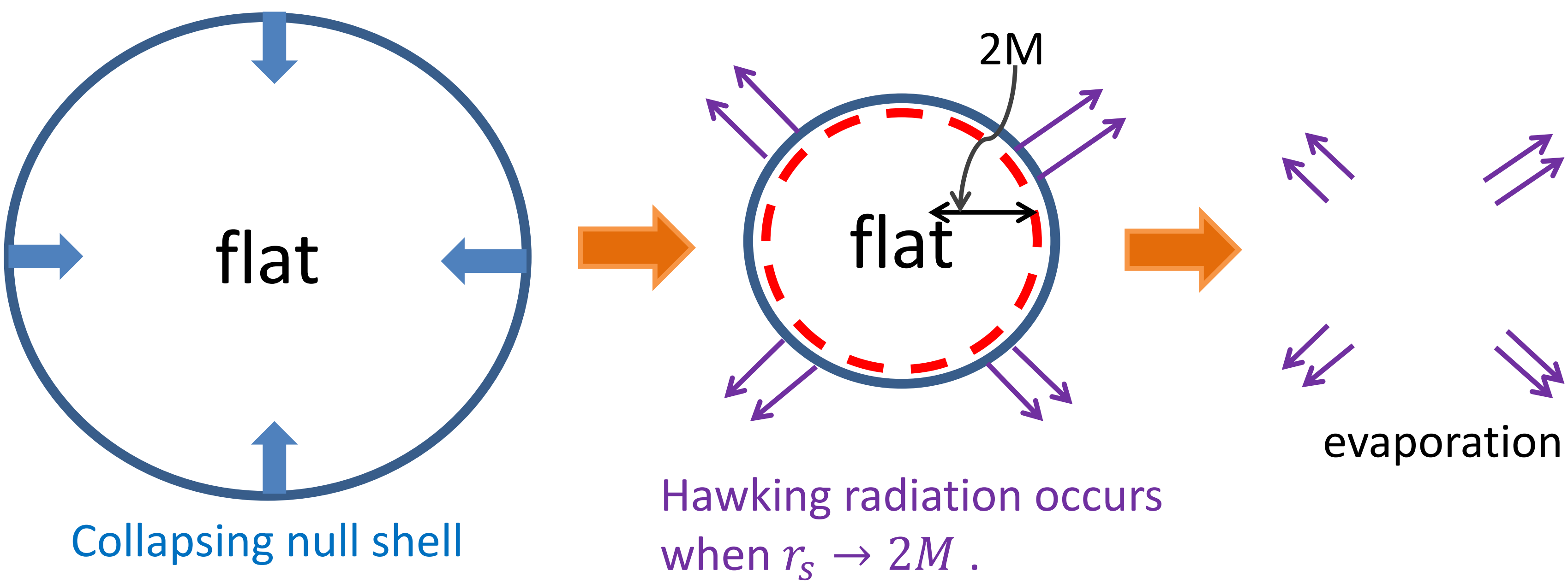
(2) The radiation fields are treated quantum mechanically on the back ground metric $g_{\mu\nu}$.
Once the metric is given, the time evolution of the fields are completely determined, and we can evaluate the expectation value of the energy momentum tensor $\langle T_{\mu\nu}\rangle$.

This is still a complicated problem, so we will try to find a self consistent solution of (1) and (2) under some approximations.

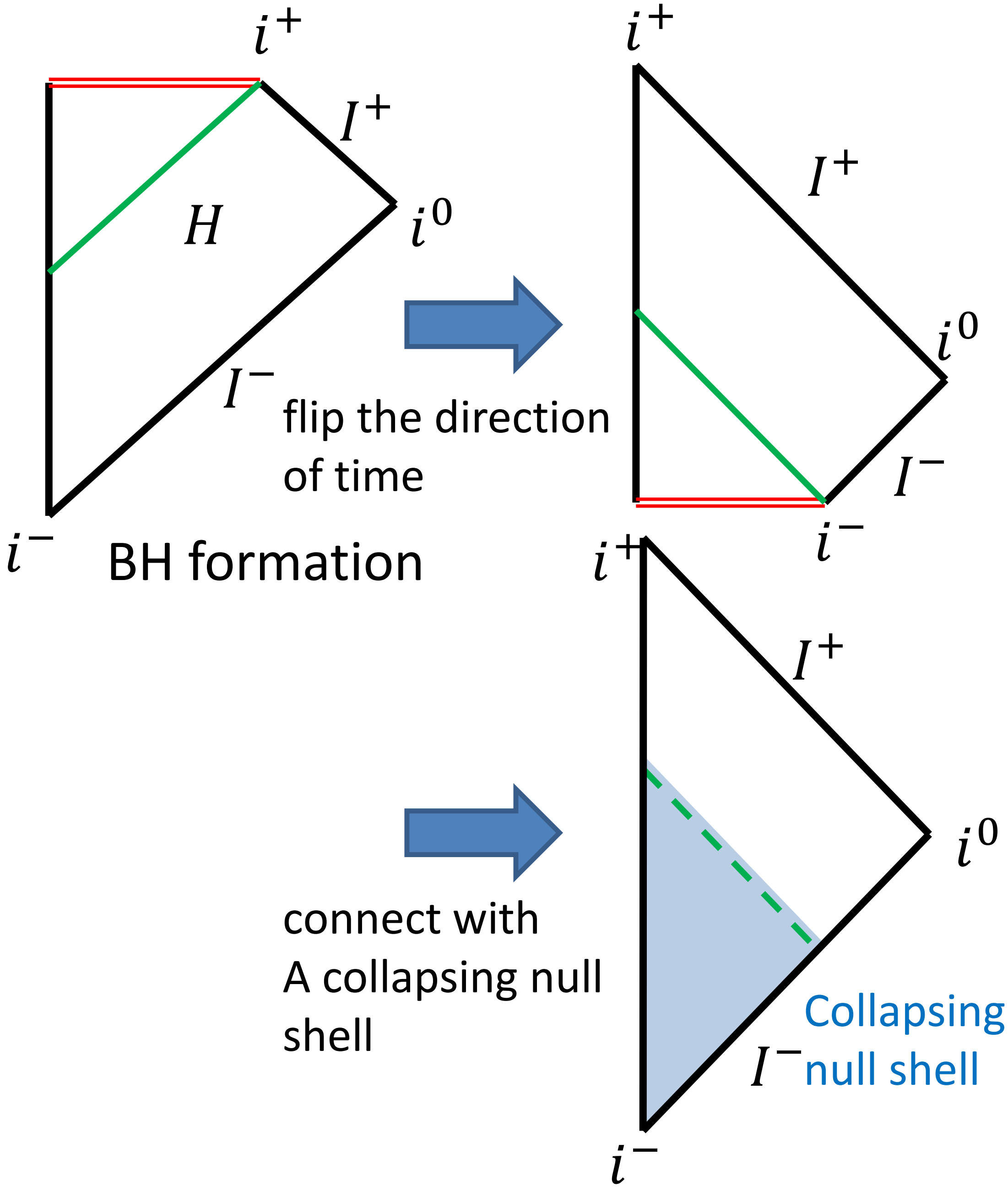
Introduction (3):
Simple minded viewpoint of the outside observer

- To be concrete, we first consider a collapsing null shell.

- If we consider only the classical mechanics, it takes infinite time for the shell to form the horizon to the outside observer.
- However, If we take the Hawking radiation into account, the BH disappears in finite time, and no horizon occurs.



Introduction (4): construction of the metric 1



In order to construct such metric, we first consider the ordinary BH formation and flip the direction of time to obtain a metric of evaporation.

In order to eliminate the past horizon we connect the metric with a collapsing null shell.

Introduction (5): construction of the metric 2

To construct a simple model, we further assume that the metric outside the null shell is given by the outgoing Vaidya metric.

So far, we have just made up a metric that looks like evaporating BH

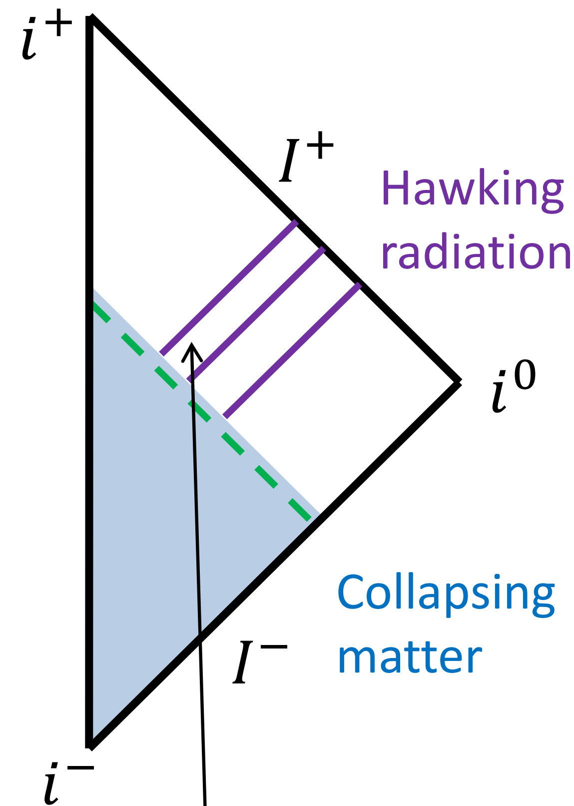
⇒Question:

Is this a solution of the self-consistent Einstein equation?

More precisely,

(1) Does Hawking radiation occurs?

(2) Does the metric indeed satisfy the Einstein equation?



Hawking radiation occurs without horizon.

The model (1)

We start with the following model:

(1) We consider a collapsing spherically symmetric null shell.
The inside is flat.

(2) We assume that the metric outside the shell is given by the Vaidya metric.

(3) We evaluate the expectation value of the energy momentum tensor $\langle T_{\mu\nu} \rangle$ by introducing the following two approximations:

(i) We keep only the S-wave for the radiation field.

(ii)We use the eikonal approximation.

(4) Then we obtain the self-consistent equation that is equivalent to $G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$.

The model (2): Outgoing Vaidya metric 1

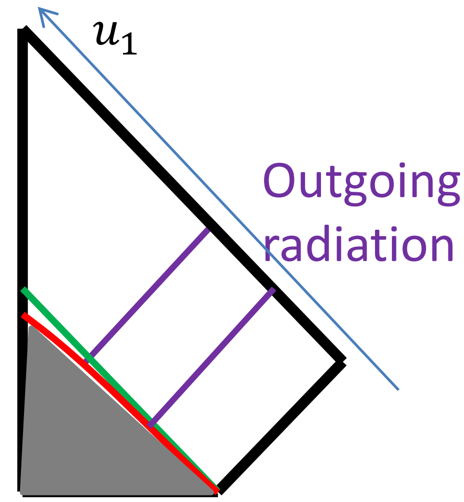
$$ds^2 = -\left(1 - \frac{a(u)}{r}\right)du^2 - 2dudr + r^2 d\Omega^2$$

r: radial coordinate

u: null coordinate "time"

$a(u)$: "Schwarzschild radius"
It can depends on *u*.

the Bondi mass $m(u) = \frac{a(u)}{2G}$

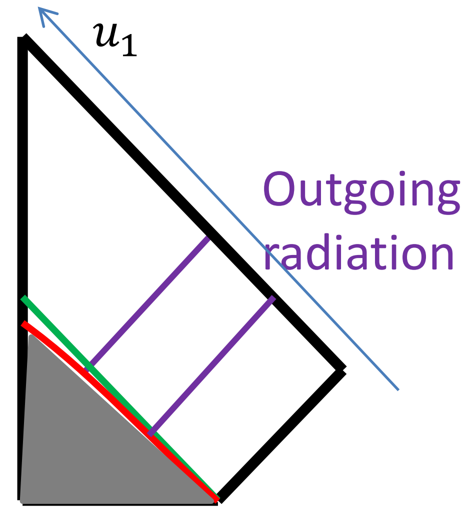


General under the assumptions:

- spherically symmetric
- Traceless : $G^{\mu}{}_{\mu} = 0$
- the only non-zero component : $G_{uu} = -\frac{\dot{a}(u)}{r^2}$

outgoing radiation

The model (3): Outgoing Vaidya metric 2



<u>Assumptions</u>

・Traceless : $G^{\mu}{}_{\mu} = 0$.

・the only non-zero component : G_{uu}

<u>The physical meaning of the assumptions</u>

- Consider emission of massless particles.
- Neglect the components $T^{\theta}{}_{\theta} = 0$.
  We consider only the S-wave of the radiation field.
- Neglect the gray-body factor.
  We neglect the reflection of the radiation by the gravitational potential. Once it is emitted, it simply flows to the infinity so that we have no incoming energy flow.

The model (4): a collapsing null shell 1

In order to make a black hole like object, we are considering a spherically symmetric collapsing null shell.

The metric inside the shell is flat

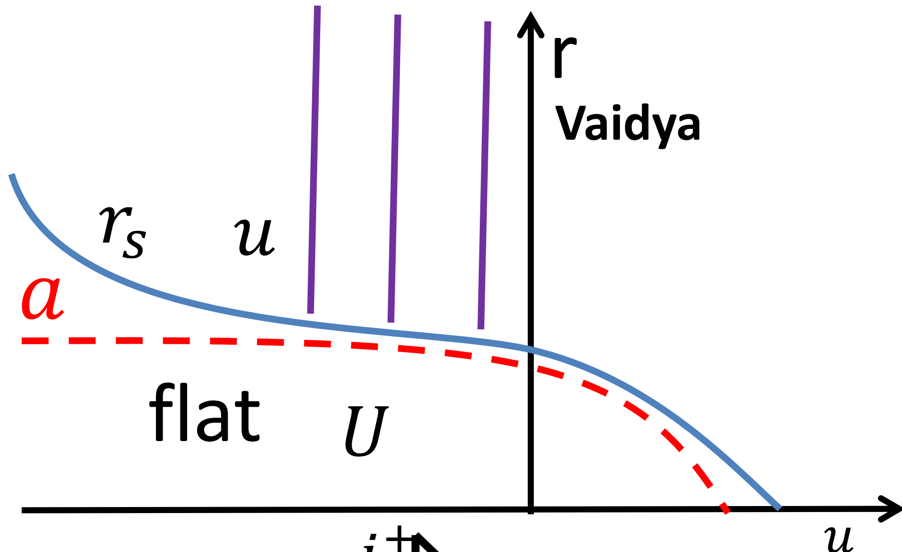
$$ds^2 = -dU^2 - 2dUdr + r^2 d\Omega^2,$$

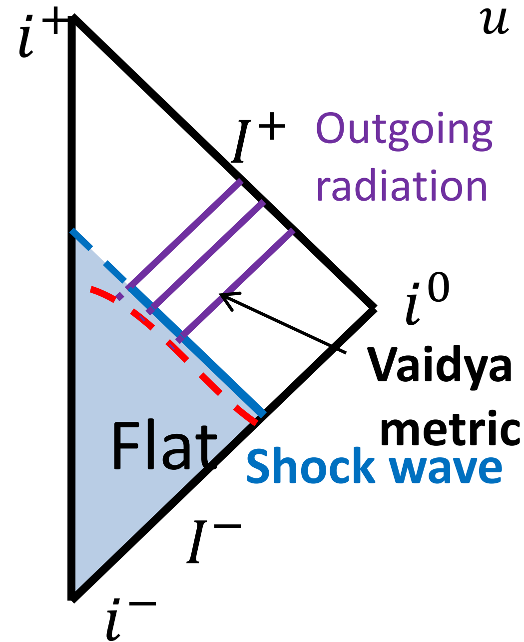
where U is the light-like coordinate inside the shell.

The metric outside the shell is assumed to be the Vaidya metric

$$ds^2 = -\left(1 - \frac{a(u)}{r}\right) du^2 - 2dudr + r^2 d\Omega^2 \ ,$$

where u is the light-like coordinate outside the shell.





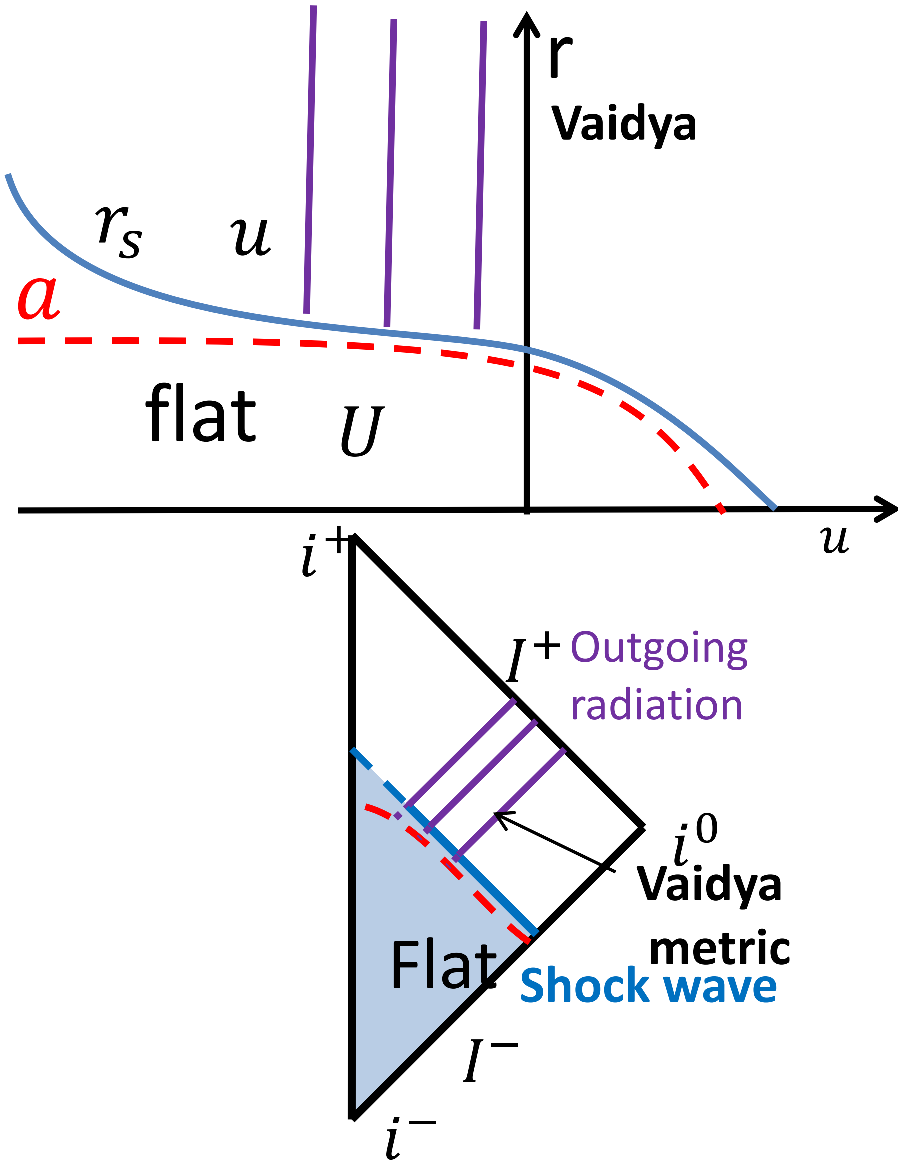
The model (5): a collapsing null shell 2

$r_s(u)$: the radius of the null shell

The junction condition of U and u is given by the fact that r_s is the locus of ingoing light for the metric of both sides:

$$\frac{r_s(u)-a(u)}{r_s(u)}du = -2dr_s = dU$$

Once $a(u)$ is given, this equation determines
(1) $r_s(u)$ as a function of u , and
(2) the relation between U and u .



The model (6): a collapsing null shell 3

The Einstein tensor $G_{\mu\nu}$

Metric ⇒ Einstein tensor

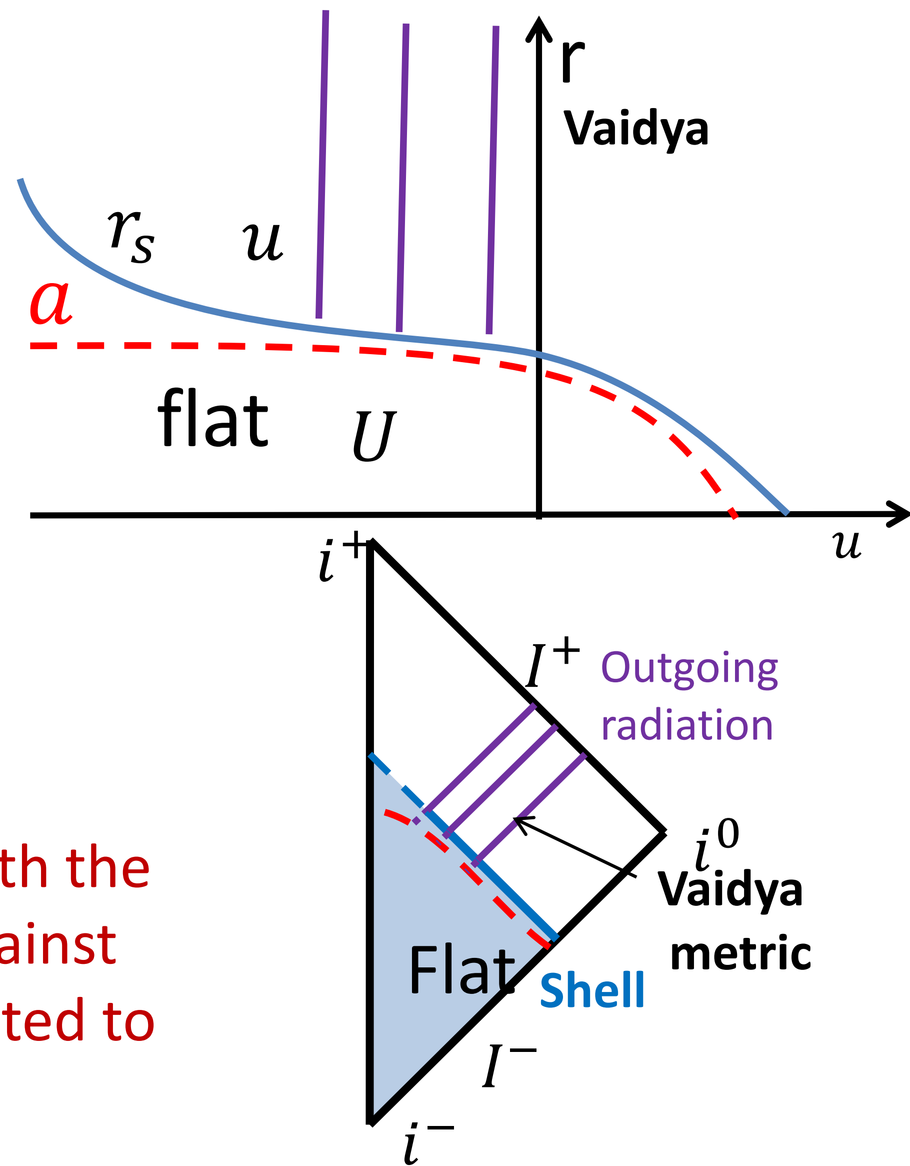
inside the shell: $G_{\mu\nu} = 0$

outside: $G_{uu}/8\pi G = \frac{-\dot{m}}{4\pi r^2}$,

$G_{others} = 0.$

$$\boldsymbol{m(u) = \frac{a(u)}{2G}}$$

The energy simply flows from the shell to infinity.



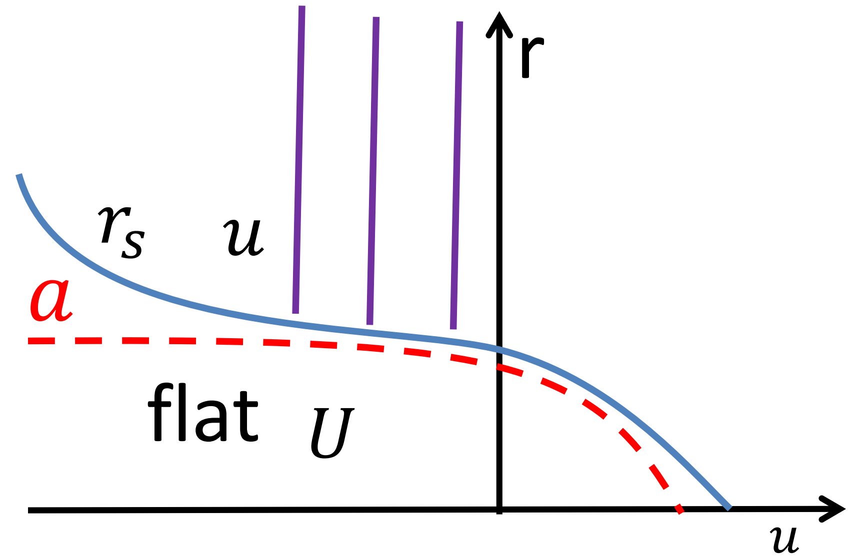
Barrabes-Israel null condition:

$$\mu = \frac{m}{4\pi {r_s}^2} \text{ and } p = \frac{-r_s \dot{m}}{2\pi (r_s - a)^2}.$$

If $\dot{a}<0$, then $p>0$. It is consistent with the picture that the shell is shrinking against the pressure and the work is converted to the radiation.

The model (7): Energy flux 1

The next step is to estimate the expectation value of the energy momentum tensor $\langle T_{\mu\nu} \rangle$ of the radiation fields in this back ground metric.



We consider a massless scalar field and introduce the following approximations:

(1) We take only the S-wave.

(2) We use the eikonal approximation.

Then the only nonzero component is $\langle T_{uu} \rangle$, and the total energy flux $J(u) \equiv 4\pi r^2 \langle T_{uu} \rangle$ is given by the Schwarzian derivative:

$$\boxed{J(u) = \frac{\hbar}{8\pi}\left[\frac{\ddot{U}(u)^2}{\dot{U}(u)^2} - \frac{2\dddot{U}(u)}{3\dot{U}(u)}\right] \equiv \frac{\hbar}{8\pi}\{u, U\}}$$

The model (8): Energy flux 2

<u>Derivation of the flux formula 1</u>

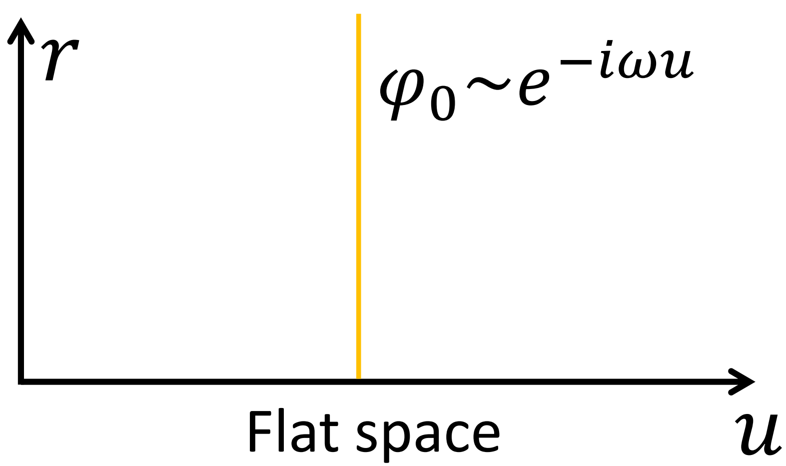
If we consider only the S-wave, in the flat space we have

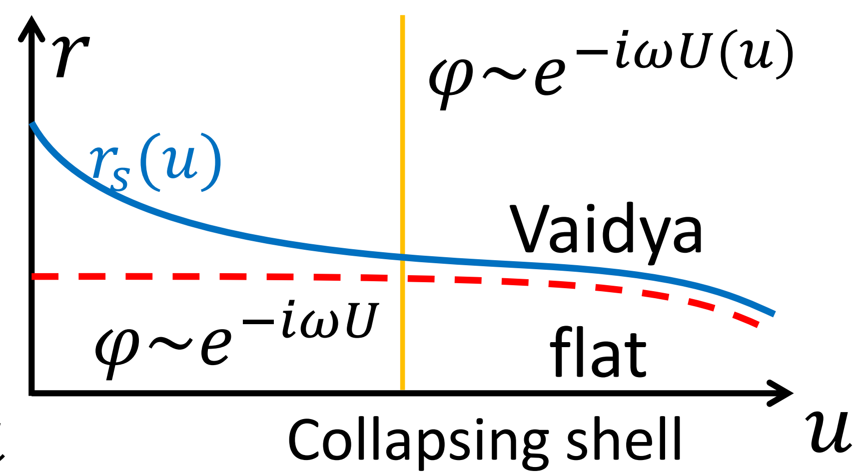
$$\phi_0(u) = \int_0^\infty \frac{d\omega}{2\pi}\left(\frac{e^{-i\omega u}}{\sqrt{4\pi\omega}r}a_\omega + \frac{e^{i\omega u}}{\sqrt{4\pi\omega}r}{a_\omega}^\dagger\right).$$

The vacuum is given by $a_\omega|0>=0$.

If we use the eikonal approximation, in our metric we have

$$\phi(u) = \int_0^\infty \frac{d\omega}{2\pi}\left(\frac{e^{-i\omega U(u)}}{\sqrt{4\pi\omega}r}a_\omega + \frac{e^{i\omega U(u)}}{\sqrt{4\pi\omega}r}{a_\omega}^\dagger\right).$$





The model (9): Energy flux 3

Derivation of the flux formula 2

Then the expectation value of the energy momentum tensor is calculated by the point splitting method:

$$
\begin{aligned}
&< 0|:T_{uu}(u):|0 > \\
&= \lim_{u'\to u}[< 0|:\partial_u\phi(u)\partial_u\phi(u'):|0 > - < 0|:\partial_u\phi_0(u)\partial_u\phi_0(u'):|0 >] \\
&= \frac{\hbar}{4\pi r^2}\lim_{\varepsilon\to 0}\int_0^{2\pi}\frac{d\omega}{2\pi}\omega\dot{U}\,(u+\varepsilon)\dot{U}\,(u-\varepsilon)e^{-i\omega\left(U(u+\varepsilon)-U(u-\varepsilon)\right)} - (U=u) \\
&= \frac{1}{4\pi r^2}\frac{\hbar}{8\pi}\{u,U\}
\end{aligned}
$$

In this approximation, the other components are zero because the eikonal depends only on u :

$< 0|:T_{others}(u):|0 >=0.$

The model (10) : self-consistent equations 1

(1) We have considered the metric obtained by connecting Vaidya metric and the flat metric on a collapsing null shell.
The only nonzero component of the Einstein tensor is

$$G_{uu}/8\pi G = \frac{-\dot{m}}{4\pi r^2}. \qquad \boldsymbol{m(u) = \frac{a(u)}{2G}}$$

(2) We have evaluated the expectation value of the energy momentum tensor by introducing two approximations: (i) S-wave, (ii) eikonal approximation.
The only nonzero component is

$$< 0|{:}T_{uu}(u){:}|0 > = \frac{1}{4\pi r^2}\frac{\hbar}{8\pi}\{u, U\}.$$

Within these approximations, the self consistency equation $G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle$ becomes the energy conservation: $\dot{m} = -J(\mathrm{u})$, or equivalently,

$$\frac{da}{du} = -\frac{{l_p}^2}{4\pi}\{u, U\}. \qquad {l_p}^2 = \hbar G$$

The model (11) : self-consistent equations 2

As we have seen, U and r_s are determined by the Junction condition:

$$U(u) = -2r_s(u), \quad \frac{dr_s}{du} = -\frac{r_s(u)-a(u)}{2r_s(u)}.$$

Thus we have the following coupled equations for $a(u)$ and $r_s(u)$:

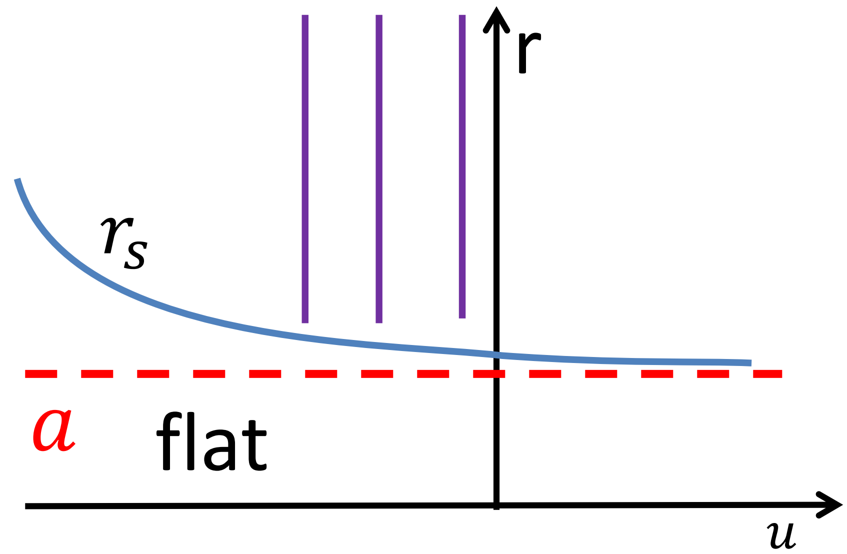
$$\frac{da}{du} = -\frac{{l_p}^2}{2\pi}\{u, r_s\}$$
$$\frac{dr_s}{du} = -\frac{r_s - a}{2r_s}$$

The r.h.s. has higher derivative.
⇒Singular perturbation

Test of the Energy flux formula

In order to test the formula, let us consider a collapsing null shell in the Schwarzschild metric.
Here, we do not take the back reaction from the radiation into account.
Then r_s is determined by

$$\frac{dr_s}{du} = -\frac{r_s(u)-a}{2r_s(u)}.$$



For large values of u the solution is given by $r_s(u) \sim a + Ce^{-u/2a}$,
and the energy flux is

$$J(u) = \frac{\hbar}{8\pi}\{u, r_s\} = \frac{\hbar}{8\pi}\left[\frac{\ddot{r}_s(u)^2}{\dot{r}_s(u)^2} - \frac{2\dddot{r}_s(u)}{3\dot{r}_s\,(u)}\right] = \frac{\hbar}{96\pi}\frac{1}{a^2},$$

which is consistent with the Hawking radiation.
We can check that the energy spectrum is indeed the Planck distribution of temperature $T = \frac{\hbar}{4\pi a}$.

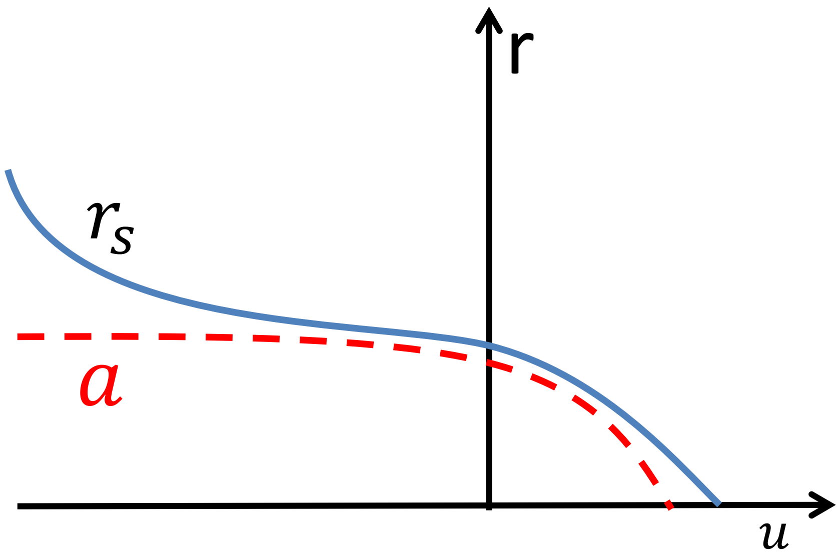
The radius of the shell r_s

The radius r_s is given by

$$\frac{dr_s}{du} = -\frac{r_s(u) - a(u)}{2r_s(u)}.$$

If a varies slowly as a function of u, $|\dot{a}| \ll 1$, roughly speaking, r_s is chasing a in the time scale $\sim 2a$. Therefore one expects for large u

$$\boldsymbol{r_s(u) \approx a(u) + Ce^{-\frac{u}{2a(u)}}}.$$

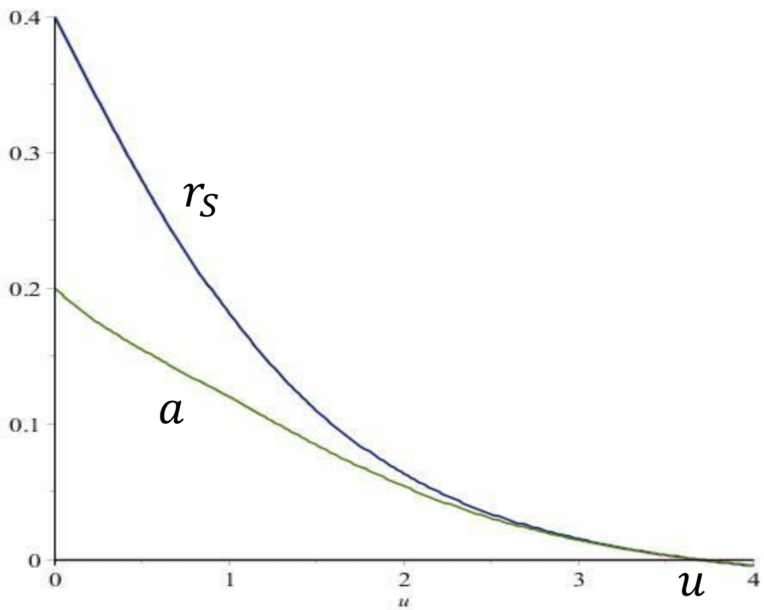


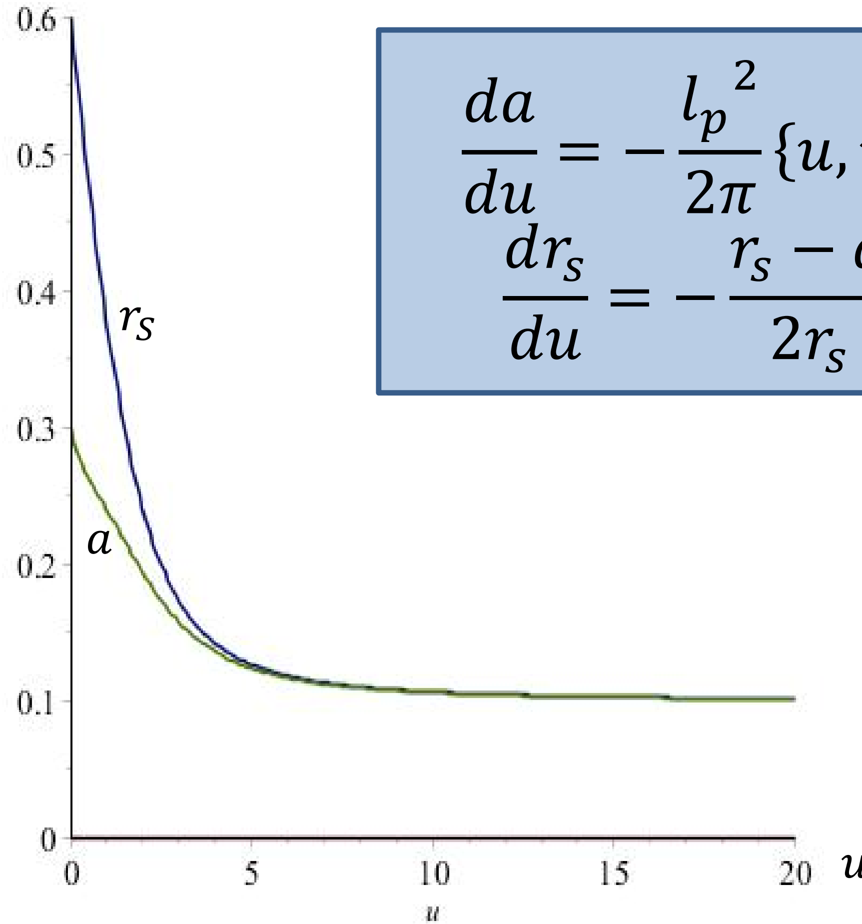
However, a is running away from r_s . When r_s comes close to the former position of a in time $2a$, a moves about $2a\dot{a}$. Therefore the asymptotic behavior of r_s is given by

$$\boldsymbol{r_s(u) \approx a - 2a\dot{a} + Ce^{-\frac{u}{2a}} \approx R(a)}.$$

This can be checked in a more rigorous manner.

Numerical Results for a single collapsing shell





$$\frac{da}{du} = -\frac{{l_p}^2}{2\pi}\{u, r_s\}$$

$$\frac{dr_s}{du} = -\frac{r_s - a}{2r_s}$$

The object evaporates in a finite time.

Radiation arises for a while, but stops eventually.

Why does the evaporation stop?

For large u ,

$$r_s(u) \approx a(u) - 2a(u)\dot{a}(u)$$

If $\dot{a}(u)$ is very small, we can approximate

$$r_s(u) \approx a(u).$$

$$\frac{da}{du} = -\frac{{l_p}^2}{4\pi}\{u, r_s\}$$
$$\frac{dr_s}{du} = -\frac{r_s - a}{2r_s}$$

Then we can solve the first equation as

$$u = u_0 + \frac{e^{-\frac{D^2}{2}}}{6\pi B}\int_D^{\xi} d\xi'\, e^{\frac{1}{4}\xi'^2}\ ,$$

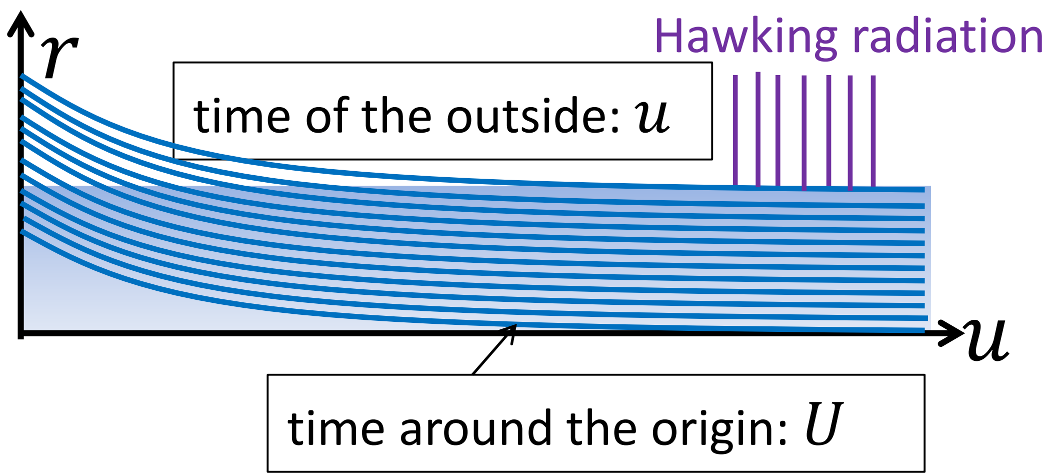
$$a(u) = a(0) - B\int_D^{\xi} d\xi'\, e^{-\frac{1}{4}\xi'^2}\ ,$$

where u_0, B, D are integration constants.

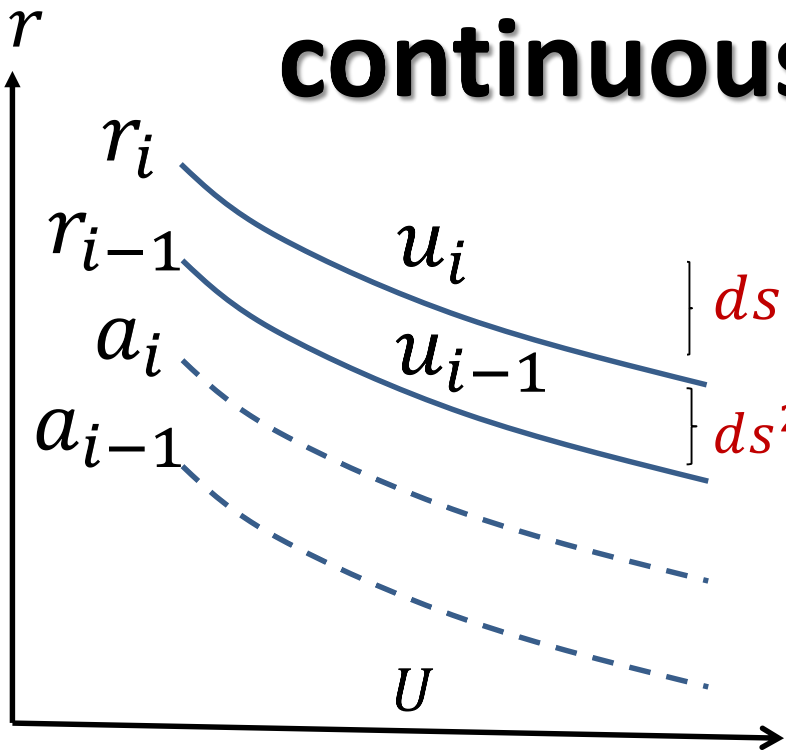
$a(u)$ does not necessarily vanish as $u \to \infty$.

continuous distribution (1)

We consider a continuously distributed concentric null shell collapsing to the origin. Each shell is approaching to its own Schwarzschild radius.



continuous distribution (2)



$$ds^2 = -\left(1 - \frac{a_i}{r}\right) du_i^2 - 2du_i dr + r^2 d\Omega^2$$

$$ds^2 = -\left(1 - \frac{a_{i-1}}{r}\right) du_{i-1}^2 - 2du_{i-1} dr + r^2 d\Omega^2$$

$$a_i = 2Gm_i \gg l_p$$

$$(i = 1,2, .., n)$$

The junction condition

$$\frac{r_i - a_i}{r_i} du_i = -2dr_i = \frac{r_i - a_{i-1}}{r_i} du_{i-1} \qquad a_0 = 0,\ u_0 = U.$$

$$\Rightarrow \frac{dr_i}{du_i} = -\frac{r_i - a_i}{2r_i}, \quad \frac{du_i}{du_{i-1}} = \frac{r_i - a_{i-1}}{r_i - a_i} = 1 + \frac{a_i - a_{i-1}}{r_i - a_i}.$$

Einstein eq.

$$\frac{da_i}{du_i} = -\frac{N l_p{}^2}{4\pi} \{u_i, U\}.$$

N the degrees of freedom of the fields. (e.g. $N \sim 100$)

continuous distribution (3)

The coupled equation is solved by the following Ansatz:

$$\frac{da_i}{du_i} = -C\frac{1}{a_i^2}$$ ⇔ The inside of each shell looks like the ordinary evaporating BH.

$$r_i = a_i - 2a_i\frac{da_i}{du_i} = a_i + \frac{2C}{a_i}$$ ⇔ Each shell has become stationary.

continuous distribution (4)

Formula: $\{u_i, U\} = \frac{1}{3}\left(\frac{d\eta_i}{du_i}\right)^2 - \frac{2}{3}\frac{d^2\eta_i}{du_i^2}$, $\eta_i = \log\frac{dU}{du_i}$.

From the Ansatz, we have

$$\eta_i - \eta_{i-1} = \log\frac{\frac{dU}{du_i}}{\frac{dU}{du_{i-1}}} = -\log\frac{du_{i-1}}{du_i} = -\log(1 + \frac{a_i - a_{i-1}}{r_i - a_i})$$

$$\approx -\frac{a_i - a_{i-1}}{r_i - a_i} = -\frac{(a_i - a_{i-1})}{\frac{2C}{a_i}} \approx -\frac{1}{4C}(a_i^2 - a_{i-1}^2) .$$

With the boundary conditions $\eta_0 = a_0 = 0$, we obtain

$$\eta_i = -\frac{1}{4C}a_i^2 \quad \Rightarrow \frac{d\eta_i}{du_i} = -\frac{1}{2C}a_i\frac{da_i}{du_i} = \frac{1}{2a_i} .$$

Then we have

$$\{u_i, U\} \approx \frac{1}{12}\frac{1}{a_i^2} ,$$

which indicates

$$C = \frac{Nl_p^2}{48\pi} .$$

Perturbation around the solution

Parametrize the deviations from the solution as

$$a_i = a_i^{(0)}(1 + A_i),\ r_i = r_i^{(0)}(1 + R_i),\ \eta_i = \eta_i^{(0)}(1 + H_i)\,,$$

and rescale the local time as

$$u_i^{(0)} = a_i^{(0)} t_i\ .$$

Then the linearized equation becomes

$$\frac{dR_i}{dt_i} = A_i - R_i + \frac{1}{4} H_i$$

$$\frac{2}{3}\left(a_i^{(0)}\right)^2 \frac{d^2 H_i}{dt_i^2} + 4C\frac{dA_i}{dt_i} = -CH_i$$

$$c_i^{(0)} H_i - c_i^{(0)-1} H_{i-1} + 2\left(c_i^{(0)} A_i - A_{i-1}\right) = \frac{1}{C}\left(a_i^{(0)}\right)^2 \ (c_i^{(0)} - 1)(R_i - A_i)\,,$$

where $c_i^{(0)} = \frac{a_i^{(0)}}{a_{i-1}^{(0)}}$.

For $a_i^{(0)} \gg \sqrt{C}$, the 3rd eq gives $R_i = A_i$, and the 1st and 2nd eqs become

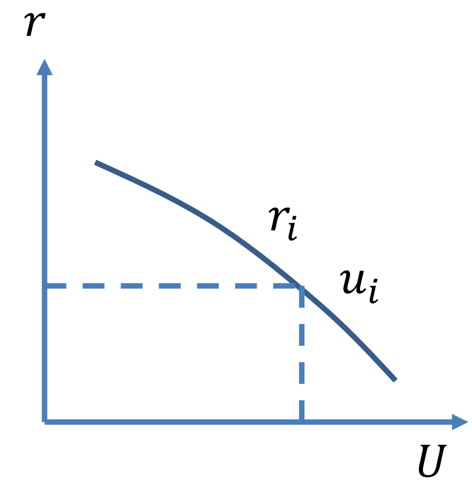
$$\left(a_i^{(0)}\right)^2 \frac{d^2 H_i}{dt_i^2} = -3CH_i \qquad \Rightarrow \text{ marginally (un)stable.}$$

continuous distribution (5): the inside metric

If we consider the shell which passes r at time U,

the metric just outside the shell is given by

$$ds^2 = -\left(1 - \frac{a_i}{r}\right) du_i^2 - 2du_i dr + r^2 d\Omega^2 \,.$$



Using $$1 - \frac{a_i}{r} = \frac{r_i - a_i}{r_i} = \frac{\frac{2C}{a_i}}{r_i} \approx \frac{2C}{r^2} \,,$$

$$\frac{du_i}{dU} = \exp(-\eta_i) = \exp\left(\frac{1}{4C} a_i^2\right) \approx \exp\left(\frac{1}{4C} r^2\right),$$

we have

$$C = \frac{N l_p^2}{48\pi}$$

$$ds^2 = -\frac{N{l_p}^2}{24\pi r^2} e^{\frac{24\pi}{N{l_p}^2} r^2} dU^2 - 2e^{\frac{12\pi}{N{l_p}^2} r^2} dU dr + r^2 d\Omega^2 \,.$$

continuous distribution (6): evaporation 1

Each shell radiates and shrinks as

$$\frac{da'}{du'} = -\frac{N{l_p}^2}{48\pi a'^2} .$$

In particular the outermost shell obeys

$$\frac{da}{du} = -\frac{N{l_p}^2}{48\pi a^2} .$$

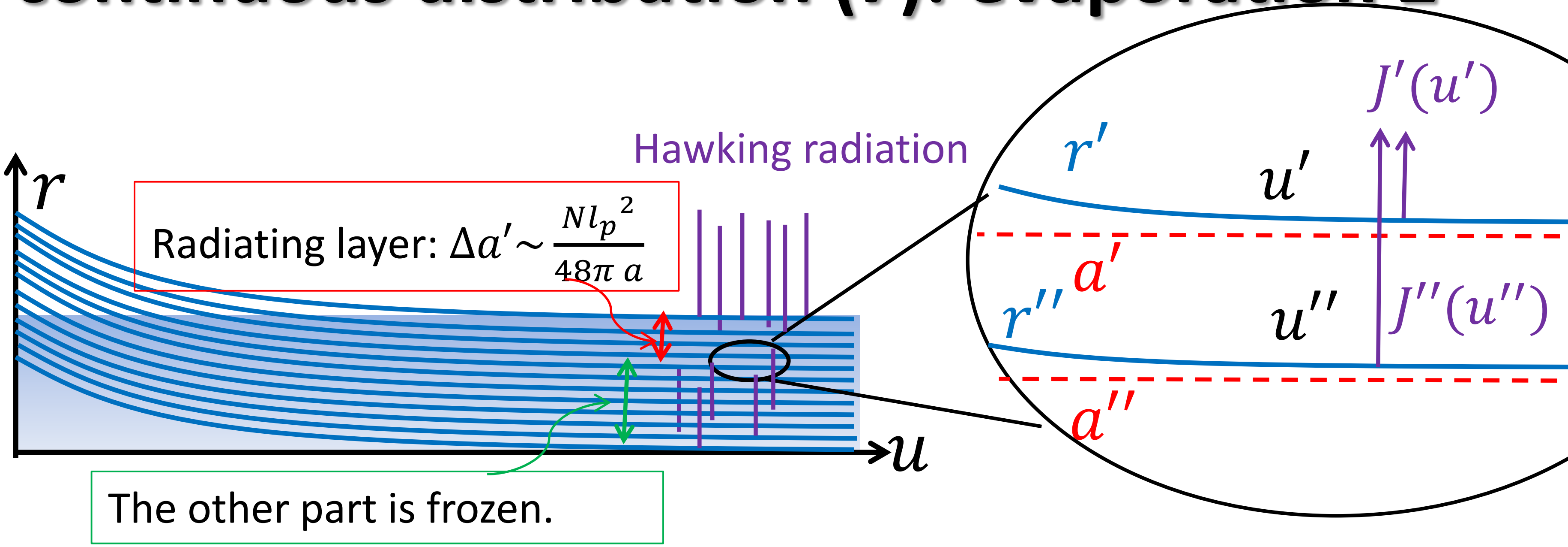
However to the outside observer, most of the radiation comes from the outermost region because of the large redshift.

The redshift can be evaluated explicitly as

$$\frac{du}{du'} = \exp\left(\frac{12\pi}{N{l_p}^2}(a^2 - a'^2)\right) \sim \exp\left(\frac{24\pi a}{N{l_p}^2}\Delta a\right).$$

Only the thin layer of width $\frac{N{l_p}^2}{24\pi\, a}$ contributes to the radiation.

continuous distribution (7): evaporation 2



The time evolution of the total mass is determined by

$$\frac{da}{du} = -\frac{N l_p{}^2}{48\pi a^2},$$

which completely agrees with the conventional BH.

It loses energy gradually from the outermost shell as if one peels off an onion.

continuous distribution (8): stationary metric 1

If we introduce the time coordinate around the origin *T* by

$$dT = dU + \frac{24\pi r^2}{N{l_p}^2} e^{-\frac{12\pi}{N{l_p}^2} r^2} dr,$$

the metric becomes

$$ds^2 = -\frac{N{l_p}^2}{24\pi r^2} e^{\frac{24\pi}{N{l_p}^2} r^2} dT^2 + \frac{24\pi r^2}{N{l_p}^2} dr^2 + r^2 d\Omega^2 \ .$$

A remarkable point is that this metric is static.
Although each shell is shrinking, the inside metric is static.

continuous distribution (9): stationary metric 2

Therefore, if we put the system in the heat bath of an appropriate temperature, it becomes completely stationary.

By introducing the time coordinate at infinity t, the inside metric is written as

$$ds^2 = -\frac{Nl_p{}^2}{24\pi r^2} e^{-\frac{24\pi}{Nl_p{}^2}(a^2-r^2)} dt^2 + \frac{24\pi r^2}{Nl_p{}^2} dr^2 + r^2 d\Omega^2 \ .$$

Remarks

(1) This is smoothly connected to the Schwarzschild metric at

$$r = a + \frac{Nl_p{}^2}{24\pi a} = R(a).$$

(2) This does not have a classical limit ($\hbar \to 0$) because it is a self-consistent solution.

(3) Classical Einstein equation is valid if
(i) $N \gg 1$, and
(ii) $r \gg \sqrt{N} l_p$.

$$\begin{aligned} R &= -\frac{48\pi}{Nl_p^2} - \frac{6}{r^2} \\ R_{\mu\nu}R^{\mu\nu} &= \frac{1152\pi^2}{N^2 l_p^4} + \frac{N^2 l_p^4}{48\pi^2 r^8} - \frac{Nl_p^2}{6\pi r^6} + \frac{16}{r^4} + \frac{288\pi}{Nl_p^2 r^2} \\ R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} &= \frac{2304\pi^2}{N^2 l_p^4} + \frac{N^2 l_p^4}{24\pi^2 r^8} - \frac{Nl_p^2}{3\pi r^6} + \frac{20}{r^4} + \frac{384\pi}{Nl_p^2 r^2} \end{aligned}$$

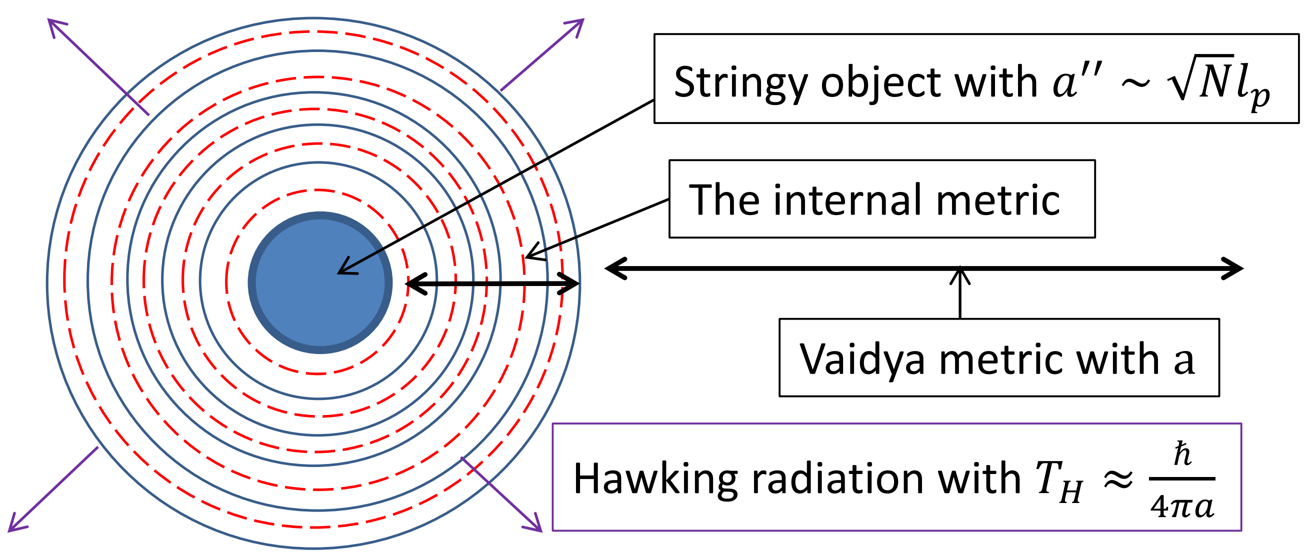
continuous distribution (10): summary

From outside, the object looks the same as the conventional BH.

In the vacuum, it evaporates as $\frac{da}{du} = -\frac{N{l_p}^2}{48\pi a^2}$.

It can be in equilibrium with a heat bath with $T = \frac{\hbar}{4\pi a}$.

If we see the inside, the entire region can be described by the semi-classical equation except for the small central region with radius of $\sim \sqrt{N}\, l_p$, which should be described by string theory,.



Relation to the Weyl anomaly (1)

- In the case of a single shell, the surface pressure results from the radiation: $p \propto \dot{a} \propto \hbar$

- In the case of continuously distributed null shell, we have a large $G^{\theta}{}_{\theta}$, and thus a large value of $G^{\mu}{}_{\mu}$:

$$G^{\mu}{}_{\mu} \sim \frac{1}{N {l_p}^2}.$$

This is consistent with the Weyl anomaly, from which we have

$$8\pi G < T^{\mu}{}_{\mu} > \sim \frac{1}{N {l_p}^2}.$$

The effect of Weyl anomaly is somehow included in the approximate solution we have obtained, although the coefficients do not match completely.

Relation to the Weyl anomaly (2): Conformal matter 1

We consider a conformal matter coupled to gravity.

From $G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle$, we have

$$G^{\mu}_{\mu} = 8\pi G\langle T^{\mu}_{\mu}\rangle = \gamma\mathcal{F} - \alpha\mathcal{G} + \frac{2}{3}\beta\nabla^2 R \quad (1)$$

where

$$\mathcal{F} \equiv C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta},$$

$$\mathcal{G} \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

$$\gamma \equiv 8\pi G\hbar c, \alpha \equiv 8\pi G\hbar a.$$

We further assume

$$\langle T^{\boldsymbol{kl}}\rangle : \langle T^{\boldsymbol{kk}}\rangle = 1 : f\ , \quad (2)$$

where $\boldsymbol{k}$ and $\boldsymbol{l}$ are ingoing and outgoing null vectors.

meaning of $f(r)$ (1)

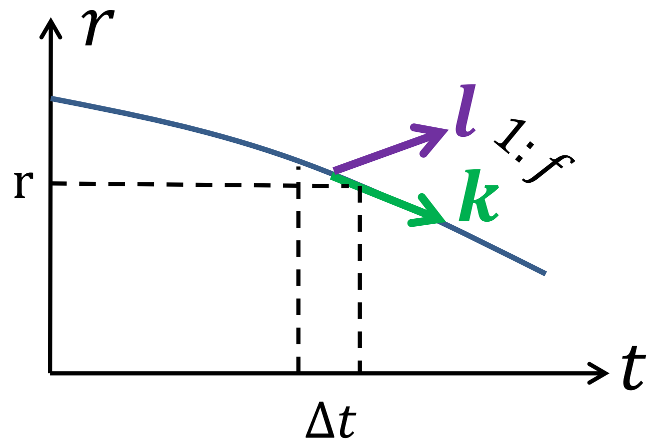
$\langle T^{\boldsymbol{k}\mu}\rangle$ is the energy momentum flow emitted from the collapsing matter.

f gives the ratio of outgoing and ingoing flow:

$$\langle T^{\boldsymbol{kl}}\rangle : \langle T^{\boldsymbol{kk}}\rangle = 1 : f.$$

$\boldsymbol{k}$: ingoing null vector

$\boldsymbol{l}$: outgoing null vector

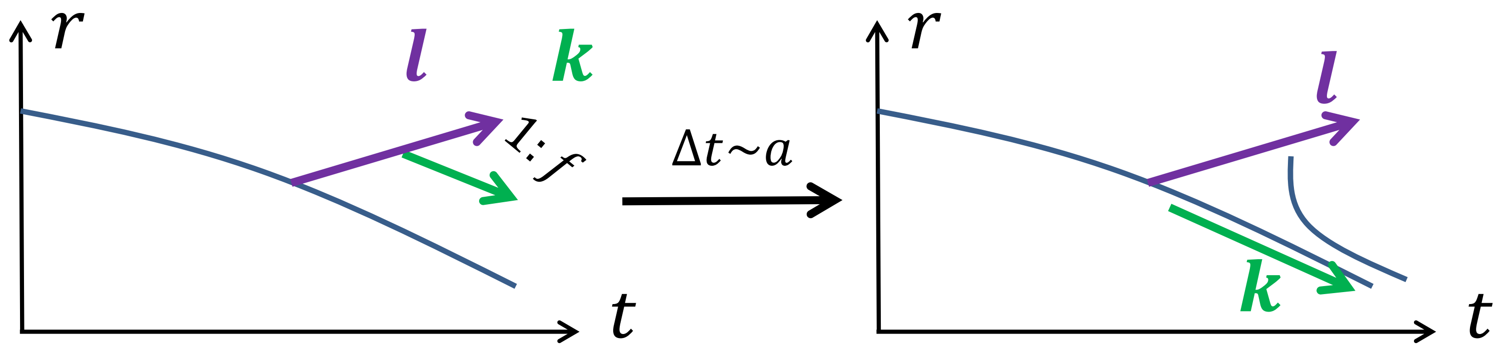


meaning of $f(r)$ (2)

If the radiated particles are all massless and emitted purely in the radial direction, $f = 0$.

f represents the effects of massive particles, reflection of the emitted particles, and the emission of particles in off radial directions.

example (reflection)



Relation to the Weyl anomaly (3): Conformal matter 2

Solving the anomaly eq.(1):

Substitute $ds^2 = -\frac{1}{B(r)} e^{A(r)} dt^2 + B(r) dr^2 + r^2 d\Omega^2$

to $G^{\mu}_{\mu} = \gamma \mathcal{F} - \alpha \mathcal{G} + \frac{2}{3} \beta \nabla^2 R$.

Assuming $A \sim B \sim r^2 / l_P^2 \gg 1$, we have

$$\frac{A'^2}{2B} + \cdots = \gamma \left[\frac{A'^4}{12B^2} + \cdots \right] - \alpha \left[-\frac{2A'^2}{r^2 B} + \cdots \right] + \frac{2}{3} \beta [\ldots]$$

$$\Rightarrow B = \frac{\gamma}{6} A'^2$$

⇒Only the c-coefficient remains.

Relation to the Weyl anomaly (4): Conformal matter 3

Solving (2):

$$\langle T^{kl}\rangle : \langle T^{kk}\rangle = 1 : f \;\Leftrightarrow \langle G^{kl}\rangle : \langle G^{kk}\rangle = 1 : f$$

$$\Leftrightarrow \; \frac{dA}{dr} = \frac{2B}{(1+f)r}$$

Assuming that $f(r)$ is a constant, we have the solution of (1) and (2)

$$B(r) = \frac{3(1+f)^2 r^2}{2\gamma}, A(r) = \frac{3(1+f)}{2\gamma} r^2.$$

If $f = 0$, this solution

$$ds^2 = -\frac{1}{B(r)} e^{A(r)} dt^2 + B(r) dr^2 + r^2 d\Omega^2$$

is reduced to the previous result

$$ds^2 = -\frac{N l_p{}^2}{24\pi r^2} e^{\frac{24\pi}{N l_p{}^2} r^2} dT^2 + \frac{24\pi r^2}{N l_p{}^2} dr^2 + r^2 d\Omega^2$$

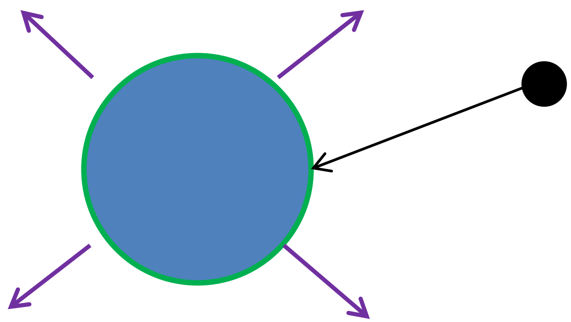
with $\frac{24\pi}{N} = \frac{3}{2c}$.

We have obtained the same result without using the eikonal or S-wave approximation.

The essence (1):
Motion of a test particle near the evaporating BH

Any object falling to an evaporating BH will never catch up with the horizon.

Consider a test particle approaching to the evaporating BH.



$$a(t) \equiv 2GM(t)$$

We assume that the outside metric is given by

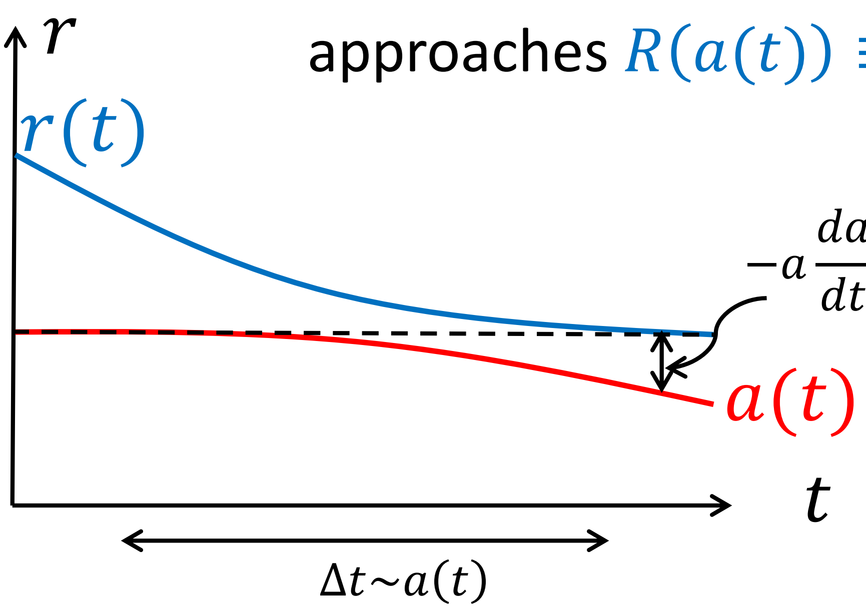
$$\boxed{\begin{aligned} ds^2 &= -\frac{r-a(t)}{r}dt^2 + \frac{r}{r-a(t)}dr^2 + r^2 d\Omega^2. \\ \frac{da}{dt} &= -\frac{2\sigma(a(t))}{a(t)^2} \end{aligned}}$$

Then, any test particle near $a(t)$ follows

$$\frac{dr(t)}{dt} = -\frac{r(t)-a(t)}{r(t)},$$

no matter what mass or angular momentum it has.

$r(t)$ does not completely catch up with $a(t)$ but approaches $R(a(t)) \equiv a(t) + \frac{2\sigma(a(t))}{a(t)}$.



The proper distance is lager than the Planck length

$$\Delta l = \sqrt{g_{rr}}\frac{2\sigma}{a} \approx \sqrt{\sigma} \sim \sqrt{N} l_p \gg l_p$$

if $N \gg 1$.

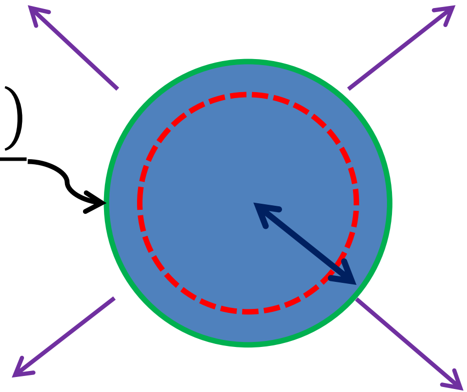
$$\frac{da}{dt} = -\frac{2\sigma(a(t))}{a(t)^2}$$

The essence (2):The boundary

- The outermost part of the matter approaches $R(a(t))$ in the time scale $\sim a$.

a boundary

$$R(a(t)) = a(t) + \frac{2\sigma(a(t))}{a(t)}$$



the outside ≒ empty.
From the outside,
the object looks the same
as the conventional BH.

Matters are stuffed inside the surface.

- the non-zero proper distance $\Delta l \sim \sqrt{\sigma}$ ⇒No horizon exists.
- We have seen that the object indeed emits the Hawking-like radiation with

$$T_H(t) = \frac{\hbar}{4\pi a(t)}.$$

Summary (1)

We tried to solve the equation $G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle$ in a self-consistent manner.

As a concrete model, we have considered a collapsing null shell, and radiation of massless fields using the following approximations:

(i) Only S-wave is taken into account.
(ii) Eikonal approximation.

Then the above equation can be solved consistently by the Vaidya metric.

The same (or more general) result is obtained by the 4-D conformal anomaly.

Summary (2)

The essential assumptions are

(1) Spherical symmetry

(2) The semi-classical approximation $G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle$.

The "BH" has a clear boundary at

$$R(a(t)) = a(t) + \frac{2\sigma\big(a(t)\big)}{a(t)}.$$

It looks the same as the conventional evaporating BH from the outside, but the interior is filled up with matters, and the metric is given by

$$ds^2 = -\frac{2\sigma(r)}{r^2} e^{-\int_r^{R(a)} dr' \frac{r'}{(1+f(r'))\sigma(r')}} dt^2 + \frac{r^2}{2\sigma(r)} dr^2 \ + r^2 d\Omega^2.$$

There is no horizon or singularity.

questions

- Precise matching of the BH entropy and the number of microstates.
- How does the information come back?
- The fate of the conserved quantities such as baryon number.
- …

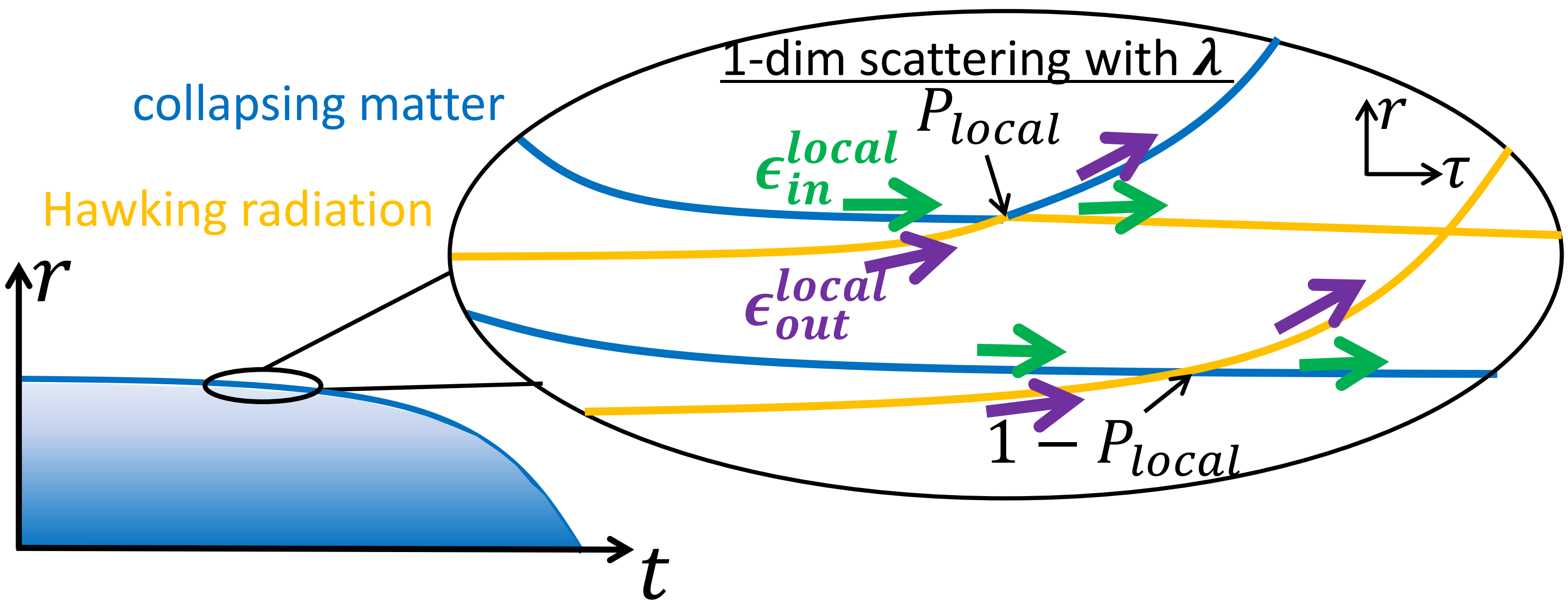
There would be many possibilities to understand the evaporation of BH, but our approach looks rather stable and robust.

information recovery

- The Hawking radiation is created near the surface.

⇒The collapsing matter and the Hawking radiation can interact.

- For simplicity, we consider only the s-wave and approximate the interaction as a one-dimensional scattering problem with a small dimensionless coupling constant λ. There are two possible cases:



- Let's estimate the scattering (proper) time scale $\Delta\tau_{scat}$:

$$\int_0^{\Delta\tau_{scat}} d\tau \, P_{local} = 1$$

The probability is estimated as follows:

- $P_{local} = \underset{\text{cross section}}{\Omega_{cross}} \times \underset{\text{number flux}}{F}$
- $\Omega_{cross} \sim \lambda^2 \frac{\hbar}{\epsilon_{in}^{local}} \frac{\hbar}{\epsilon_{out}^{local}} \sim \lambda^2 \sigma(a) e^{\frac{\tau}{\sqrt{2\sigma(a)}}}$
- $F \sim \frac{1}{4\pi a^2} \times \frac{1}{2G} \times \frac{1}{\epsilon_{out}^{local}} \sim \frac{\sqrt{\sigma(a)}}{l_p^2 a^2}$

We get

$$\boxed{\Delta t_{scat} \sim a \log \frac{l_p a}{\lambda \sigma(a)}}$$

$$J_{local}(R(a)) = \left(\frac{1}{\sqrt{-g_{tt}(R(a))}}\right)^2 \frac{\sigma(a)}{Ga^2} = \frac{1}{2G}$$

$$\epsilon_{in}^{local} \sim \frac{\hbar}{\sqrt{\sigma(a)}} e^{-\frac{\tau}{\sqrt{2\sigma(a)}}}$$

$$\epsilon_{out}^{local} \sim \frac{\hbar}{\sqrt{\sigma(a)}}$$

$$j_{local}(r) = \frac{1}{8\pi G r^2}$$

$$\int_0^{\Delta\tau_{scat}} d\tau \, P_{local} = 1$$

$$d\tau = \sqrt{-g_{tt}(r \approx R(a))} dt = \frac{\sqrt{2\sigma(a)}}{a} dt$$

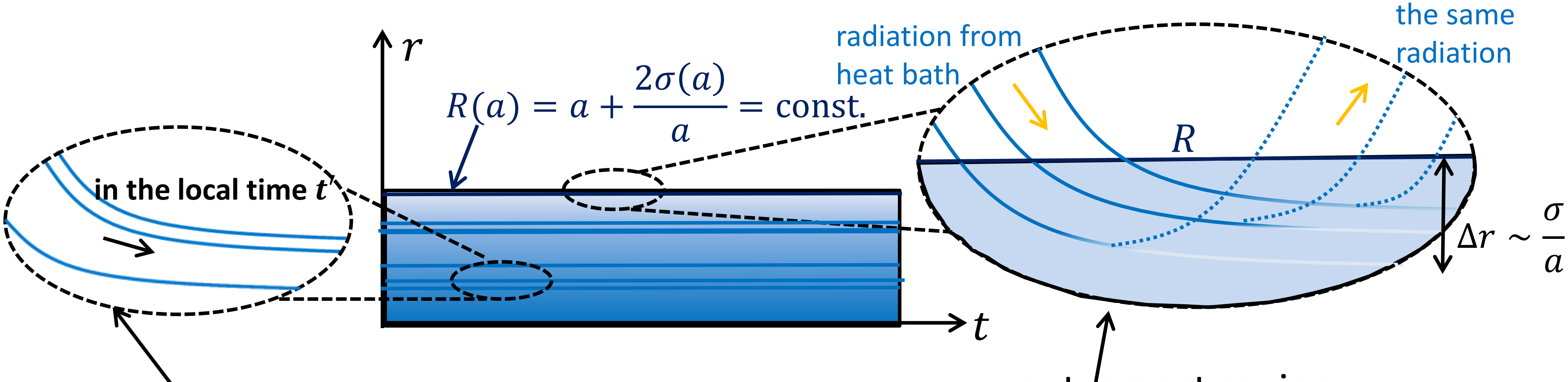
Stationary BH in the heat bath

Suppose an evaporating object with a is put in the heat bath of $T_H = \frac{\hbar}{4\pi a}$.

The matter in the outermost region comes out due to scattering.

⇒The radiation from the heat bath replaces it.

⇒The system becomes stationary.

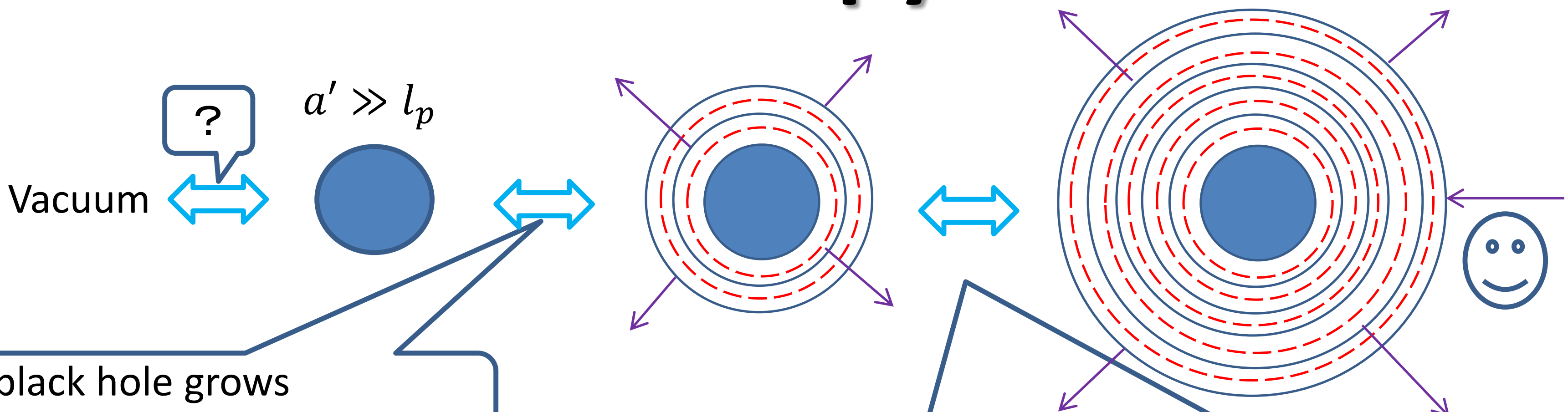


<u>deeper region</u>
The initial information is kept.

<u>outermost region</u>
The replacement occurs.

Nevertheless, the total object becomes *practically* in equilibrium with the heat bath in the sense that the outgoing and ingoing flows balance almost completely.

entropy



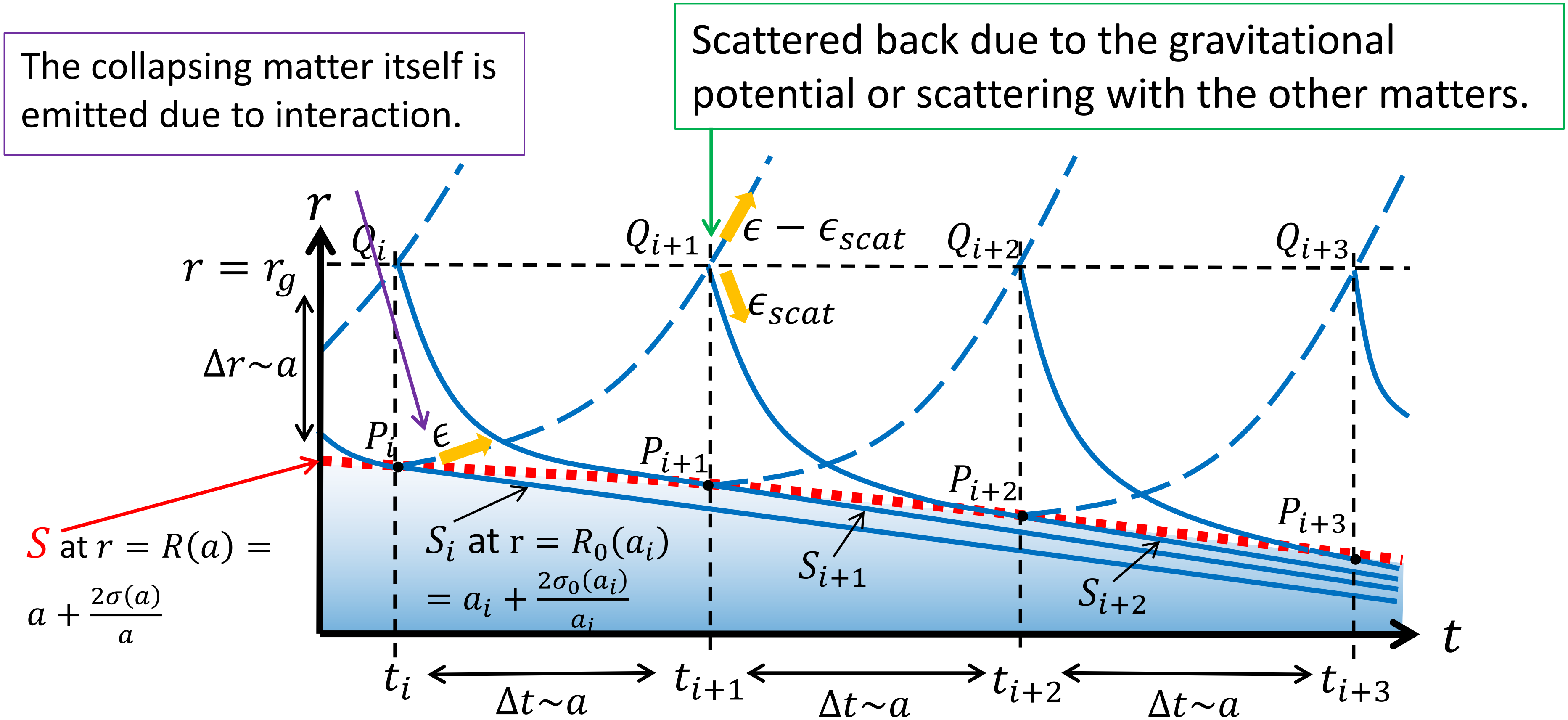
The black hole grows adiabatically in the heat bath.

We make the BH grow in the heat bath in the time scale of $\sim a^3$.
Because of the exponentially large red shift, the inside layer is almost frozen to the state when it was formed.
At each step an observer just outside the surface measures the local temperature and energy change given by the Schwarzschild metric at the surface:

$$de(r) = \frac{\mathrm{dr}}{2G\sqrt{-g_{tt}}} = \frac{\mathrm{dr}}{2G}\sqrt{\frac{24\pi}{N}\frac{r}{l_p}}, \qquad T(r) \approx \frac{\hbar}{4\pi r}\frac{1}{\sqrt{-g_{tt}}} = \frac{\hbar}{4\pi l_p}\sqrt{\frac{24\pi}{N}}$$

$\Rightarrow S_{BH} = \int_0^a \frac{de(r)}{T(r)} = \frac{A}{4{l_p}^2}$ $\Rightarrow$The entropy comes from the internal structure.

A closer look at the surface and intensity $\sigma(a)$



energy of the scattered part $\epsilon_{scat} = \frac{g(a)}{1+g(a)}\epsilon$.

$$\frac{da}{dt} = -\frac{2\sigma\big(a(t)\big)}{a(t)^2}, \qquad \sigma(a) = \frac{\sigma_0(a)}{1+g(a)}.$$

- **$g(a)$: A function that may depend on the detail of the matter**

⇒the initial-data dependence

- **$\sigma_0(a)$: raw intensity**

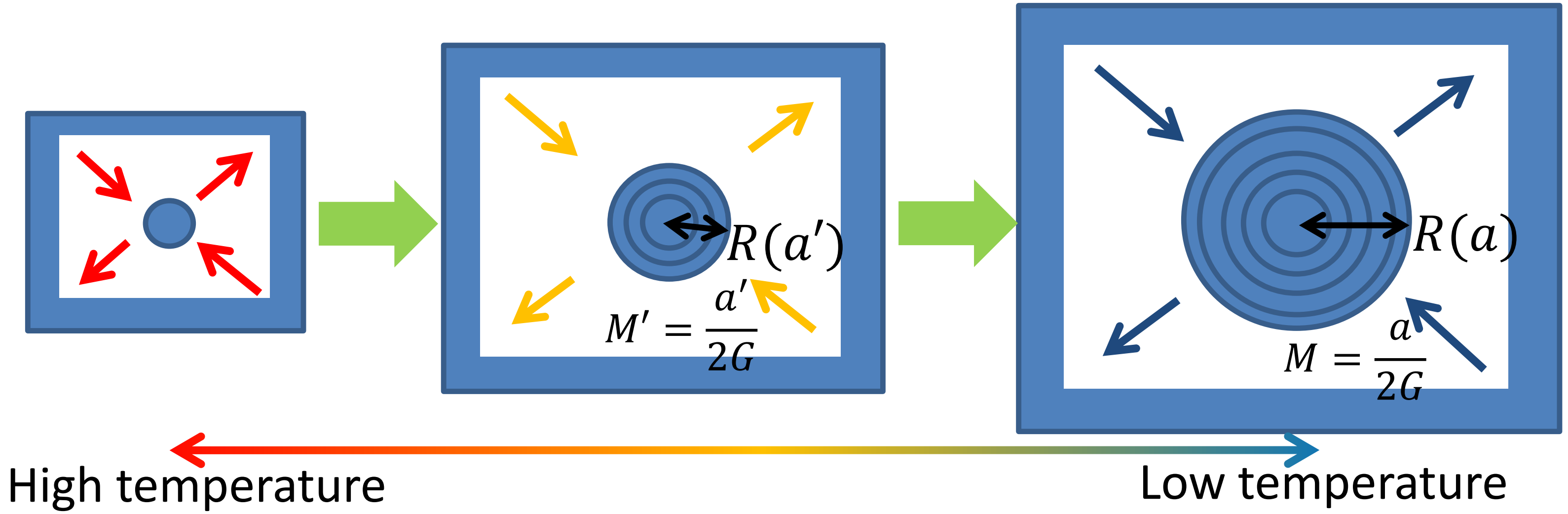
⇒The detail of the time evolution may depend on the initial data.

The adiabatically formed BHs

- Motivation and setup
- The interior metric
- Consistency checks
- Charged BH and slowly-rotating BH

Motivation and setup

- We consider a small Schwarzschild BH put in a heat bath, and grow it to a large one adiabatically by changing the temperature and size of the heat bath properly.



- surface at $r = R(a')$ + balance⇒structure at r
- spherical symmetry ⇒metric independent of the total size a.
- Let's determine the interior metric:

$$ds^2 = -\frac{1}{B(r)}e^{A(r)}dt^2 + B(r)dr^2 + r^2 d\Omega^2$$

The interior metric

The interior metric of the BH with a is

$$\boxed{ds^2 = -\frac{2\sigma(r)}{r^2} e^{-\int_r^{R(a)} dr' \frac{r'}{(1+f(r'))\sigma(r')}} dt^2 + \frac{r^2}{2\sigma(r)} dr^2 + r^2 d\Omega^2.}$$

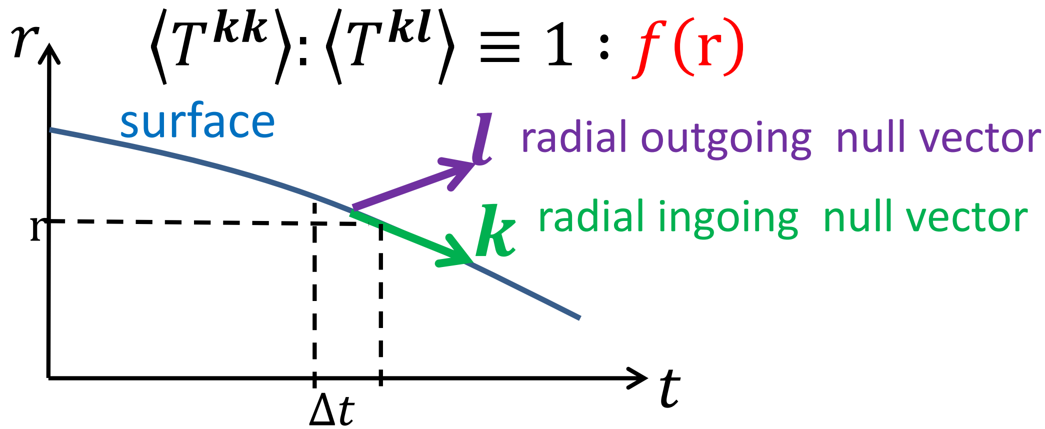
Note: The classical limit ($\hbar \to 0$) does not exist. $\sigma(r) \propto \hbar$

This is connected to the exterior metric at $r = R(a) = a + \frac{2\sigma}{a}$:

$$ds^2 = -\frac{r-a}{r} dt^2 + \frac{r}{r-a} dr^2 + r^2 d\Omega^2.$$

Two phenomenological functions:

$$\langle T^{kk} \rangle : \langle T^{kl} \rangle \equiv 1 : f(\mathrm{r})$$



$$\frac{da}{dt} = -\frac{2\sigma(a)}{a^2}$$

Consistency checks (1/2)

- Each invariance is smaller than Planck scale if $N \gg 1$:

$$R, \sqrt{R_{\alpha\beta}R^{\alpha\beta}}, \sqrt{R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}} \sim \frac{1}{(1+f)^2\sigma}$$

⇒the metric is the self-consistent solution of

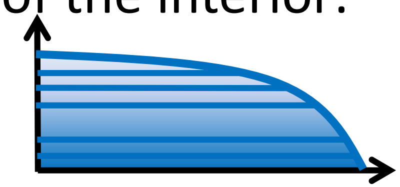
$$G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle.$$

- The energy-momentum is consistent with the general estimation:

$$-\langle T^t_t \rangle = \frac{1}{8\pi G r^2}, \quad \langle T^r_r \rangle = \frac{1}{8\pi G r^2}\frac{1-f}{1+f}, \qquad \langle T^\theta_\theta \rangle = \frac{1}{8\pi G}\frac{1}{2(1+f)^2\sigma}$$

- The area law is reproduced by summing up the entropy of the interior:

$$S = \int dl \int d\tau\, \dot{s} = \frac{A}{4 l_p^2}$$



- In the case of conformal matter, we can check $\langle T^\mu_\mu \rangle = \hbar(c\mathcal{F} - a\mathcal{G} + \frac{2}{3} b \nabla^2 R)$

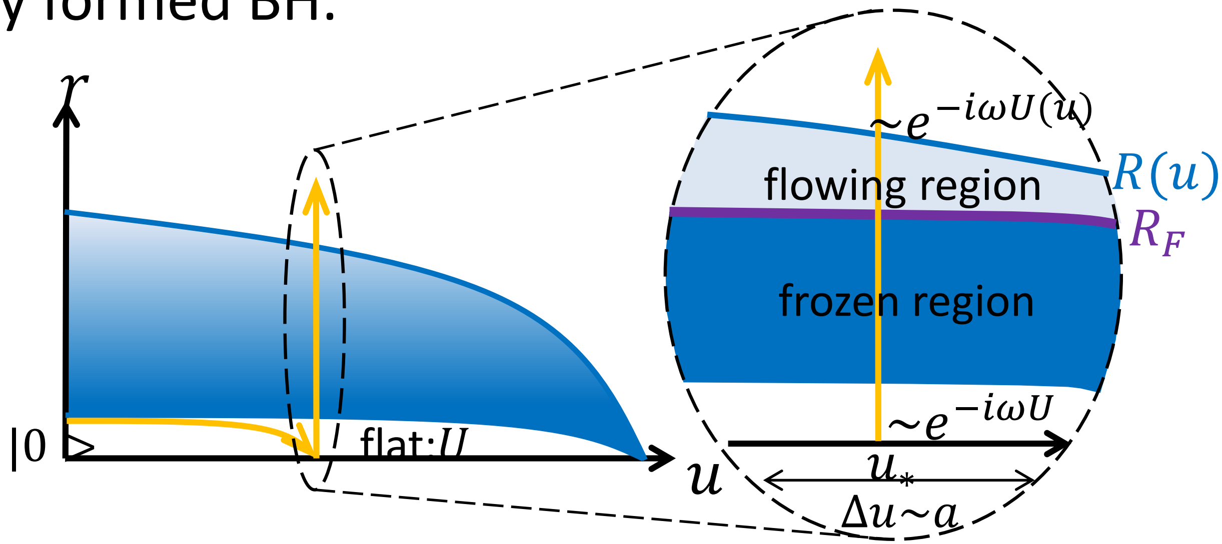
$$\sigma = \frac{8\pi l_p^2 c}{3(1+f)^2}, \qquad \langle T^\theta_\theta \rangle = \frac{1}{2}\langle T^\mu_\mu \rangle \sim \frac{1}{G l_p^2 c}$$

Consistency checks (2/2): Derivation of Hawking temperature

- In the general spacetime

$$ds^2 = -q(u,r)\left[\frac{h(u,r)}{r}du + 2dr\right]du + r^2 d\Omega^2$$

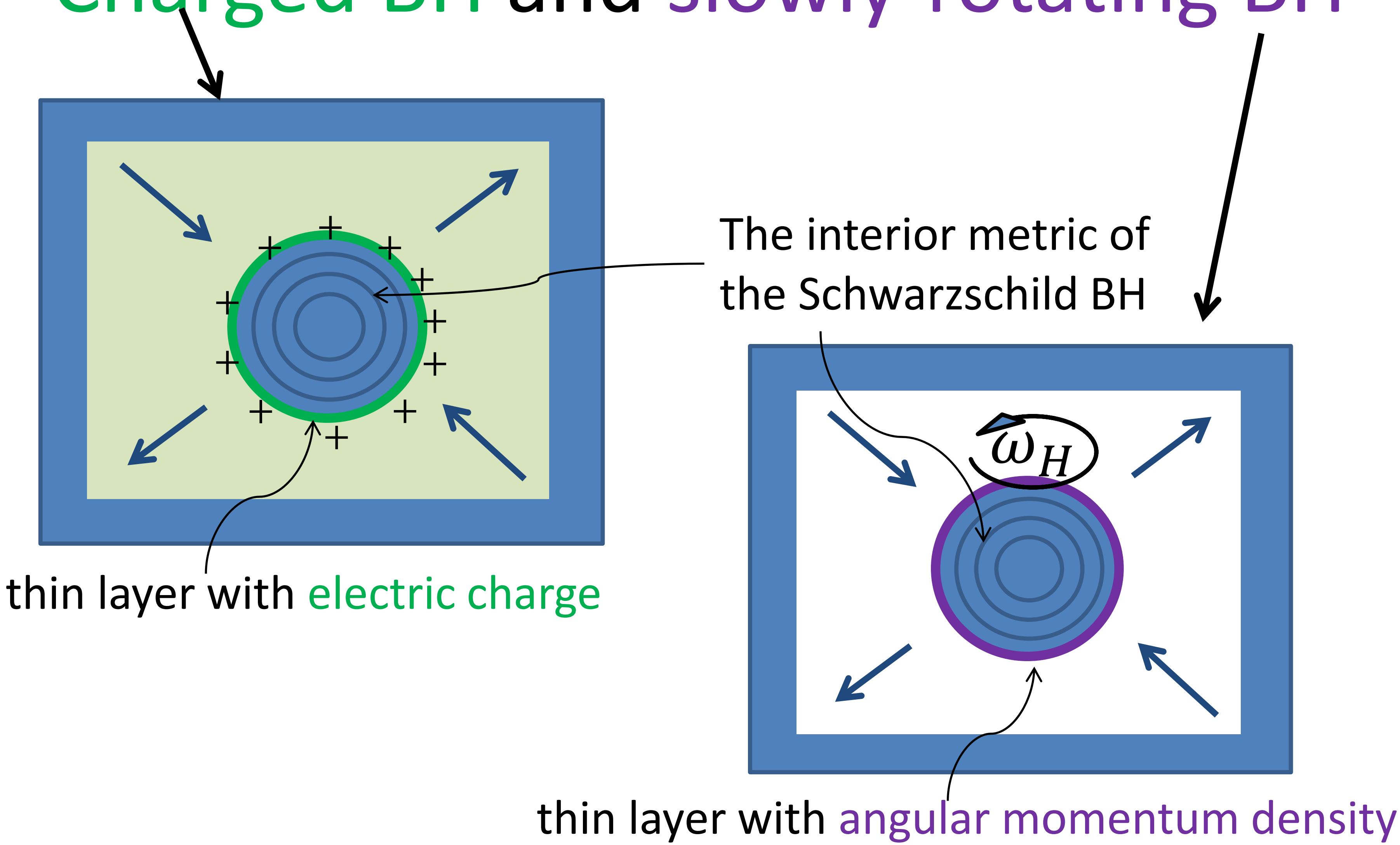
the outermost region has almost the same metric as that of the adiabatically formed BH.



- We consider the s-wave of a massless real scaler field, and solve the Heisenberg equation in the eikonal approximation.

$$\Rightarrow \langle 0|b_\omega^\dagger b_\omega|0\rangle_{u_*} = \frac{1}{e^{\hbar\omega/T_H(u_*)} - 1}, \qquad T_H(u_*) = \frac{\hbar}{4\pi a(u_*)}$$

Charged BH and slowly-rotating BH



These pictures come from a discussion based on principles of thermodynamics.